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# Computing the Scaling Exponents in Fluid Turbulence from First Principles: Demonstration of Multiscaling

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We develop a consistent closure procedure for the calculation of the scaling exponents $\zeta_n$ of the $n$th-order correlation functions in fully developed hydrodynamic turbulence, starting from first principles. The closure procedure is constructed to respect the fundamental rescaling symmetry of the Euler equation. The starting point of the procedure is an infinite hierarchy of coupled equations that are obeyed identically with respect to scaling for any set of scaling exponents $\zeta_n$. This hierarchy was discussed in detail in a recent publication by V. S. L'vov and I. Procaccia. The scaling exponents in this set of equations *cannot* be found from power counting. In this paper we present in detail the lowest nontrivial closure of this infinite set of equations, and prove that this closure leads to the determination of the scaling exponents from solvability conditions. The equations under consideration after this closure are *nonlinear* integro-differential equations, reflecting the nonlinearity of the original Navier–Stokes equations. Nevertheless they have a very special structure such that the determination of the scaling exponents requires a procedure that is very similar to the solution of *linear* homogeneous equations, in which amplitudes are determined by fitting to the boundary conditions in the space of scales. The renormalization scale that is necessary for any anomalous scaling appears at this point. The Hölder inequalities on the scaling exponents select the renormalization scale as the outer scale of turbulence $L$. We demonstrate that the solvability condition of our equations leads to non-Kolmogorov values of the scaling exponents $\zeta_n$. Finally, we show that this solutions is a first approximation in a systematic series of improving approximations for the calculation of the anomalous exponents in turbulence.

**KEY WORDS:** Turbulence; universal statistics; anomalous scaling; multifractals.

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## 1. INTRODUCTION

Unquestionably, the calculation of the scaling exponents that characterize correlation functions in turbulence is one of the coveted goals of nonlinear statistical physics. In a recent paper,[1] hereafter referred to as paper I, a formal scheme to achieve such a calculation was outlined. The scheme is non-perturbative, being based on the exact (infinite) hierarchy of coupled nonlinear equations for correlation and Green's functions. These equations respect, in the limit of infinite Reynolds numbers, the fundamental rescaling symmetry of the Euler equations. In the present paper we offer an analysis of the the lowest order closure of this hierarchy, demonstrate the existence of anomalous scaling solutions, identify the mechanism for anomalous scaling in turbulence, and discuss the series of steps available if one wants to improve the numerical values of the computed anomalous exponents. The main ideas how to achieve the lowest order closure without breaking the rescaling symmetry of the Euler equation are formulated such that they repeat essentially unchanged in any higher order closure; it seems that nothing further is required.

The scaling exponents in turbulence are defined through the scaling behavior of the structure functions of longitudinal velocity differences,

$$S_n(R) \sim R^{\zeta_n} \tag{1}$$

where $R$ is in the so-called "inertial range," and

$$S_n(R) \equiv \left\langle \left\{ [\mathbf{u}(\mathbf{r}+\mathbf{R}, t) - \mathbf{u}(\mathbf{r}, t)] \cdot \left(\frac{\mathbf{R}}{R}\right) \right\}^n \right\rangle \tag{2}$$

Here $\mathbf{u}(\mathbf{r}, t)$ is the velocity field of the fluid at a space-time point. The accumulated experimental evidence is that the numerical values of the exponents $\zeta_n$ disagree with the predictions of the classical Kolmogorov theory, according to which they are estimated as $n/3$.[2] In fact the exponents appear to depend nonlinearly on $n$, and for increasing $n$ they are considerably smaller than $n/3$. Since the Kolmogorov theory is dimensionally correct, predicting $S_n(R) \sim (\bar{\varepsilon} R)^{n/3}$, with $\bar{\varepsilon}$ the mean energy flux per unit time and mass, the experimental observations indicate the existence of a renormalization scale $L$ which is needed to nondimensionalize the corrections to the exponents,

$$S_n(R) \sim (\bar{\varepsilon} R)^{n/3} \left(\frac{L}{R}\right)^{\delta_n} \tag{3}$$

where for $n>3$ one expects $\delta_n$ positive. One role of a proper theory of anomalous scaling in turbulence is also to point out what is the renormalization scale $L$ and how it appears theoretically.

In previous work[3–5] we explained that the structure functions are not convenient objects from the point of view of theoretical analysis. They are "partly fused" objects in the sense of ref. 5, and their equations of motion contain dissipative contributions that do not disappear even in the limit of vanishing viscosity $\nu \to 0$ or infinite Reynolds number $\mathrm{Re} \to \infty$. As a result the scaling theory of these objects mixes inertial and dissipative range objects. The resulting equations in this approach are not scale-invariant, and very difficult to analyze with the aim of finding anomalous exponents. Objects whose equations of motion are scale invariant in the limit $\mathrm{Re} \to \infty$ are the "fully-unfused" space-time correlation function $\mathscr{F}_n$ that are introduced in Section 2. These equations exhibit a rescaling symmetry, and by preserving this rescaling symmetry throughout all the manipulations we are able to find new solutions of the type (3).

A central idea in the analysis is that the rescaling symmetry of the equations for the fully-unfused time correlation functions $\mathscr{F}_n$ implies their "multiscaling" representation as a weighted sum of contributions each of which is characterized by a definite scaling exponent $h$, see Eq. (21) below. The weighted sum is evaluated as a saddle-point integral determined by a scalar function $\mathscr{Z}(h)$ whose knowledge suffices to determine the scaling exponents, see Eq. (17) below. Such representations have appeared before[2, 6] in the context of "multifractals" (in which $\mathscr{Z}(h)$ is akin to these much used $f(\alpha)$ function); we stress that here they are deduced from the exact equations of motion.

We begin in Section 2 with the exact equations of motion satisfied by the space-time correlation functions in the limit of infinite Reynolds number. We then discuss the rescaling symmetry of these equations, and introduce the central idea that this chain of equations can be solved for each $h$ separately, (denoted below as a solution on a "$h$-slice"). We proceed to explain that the set of equations for the correlation functions has to be supplemented by an associated chain of equations for the Green's functions. The set of equations that we derive on an "$h$-slice" are an infinite hierarchy. If we want to find $\mathscr{Z}(h)$ we must introduce some closure of this hierarchy. This is a delicate issue, since an arbitrary closure ruins the scale invariance of the equations, and reintroduces power counting. It should be understood that any power counting leads to Kolmogorov scaling. We are seeking a new solution that cannot be obtained from power counting, and therefore must seek a closure that does not ruin the rescaling symmetry. In Section 3 we introduce the $\mathscr{Z}$-covariant closures which provide a way to search for solutions which are automatically scale invariant. In Section 4 we explain that $\mathscr{Z}(h)$ can be obtained from a solvability condition. In Section 5 we present the lowest order closure. In Section 6 we simplify the space integrals by offering a "3-scale" scalar model, and display its solution. We find $\mathscr{Z}(h)$ explicitly, and from it compute the low order scaling exponents. We demonstrate the existence of anomalous scaling, in which the scaling exponents $\zeta_n$ are neither Kolmogorov nor linear in $n$. The most important result of the present demonstration is presented graphically in Fig. 10. There one can see that the computed minimum of $\mathscr{Z}(h)+2h$ is smaller than two times the computed minimum of $\mathscr{Z}(h)+h$. As explained after Eq. (17) this means that $2\zeta_1>\zeta_2$. This is tantamount to multiscaling. The fact that $\mathscr{Z}(h)$ is positive selects the renormalization scale $L$ as the outer scale of turbulence. In Section 8 we summarize the paper and discuss the road ahead. Also in Section 8 we discuss the differences and similarities to the anomalous scaling found in the Kraichnan model of passive scalar advection.(7)

## 2. PRELIMINARY DISCUSSION AND EXACT EQUATIONS OF MOTION

To set up the development of this paper, we present now a short review of some essential ideas and results. Firstly we stress that a dynamical theory of the statistics of turbulence calls for a convenient transformation of variables that removes the effects of kinematic sweeping. In our work we use the Belinicher–L'vov transformation(8) in which the field $\mathbf{v}(\mathbf{r}_0, t_0 \mid \mathbf{r}, t)$ is defined in terms of the Eulerian velocity $\mathbf{u}(\mathbf{r}, t)$:

$$\mathbf{v}(\mathbf{r}_0, t_0 \mid \mathbf{r}, t) \equiv \mathbf{u}[\mathbf{r}+\boldsymbol{\rho}(\mathbf{r}_0, t), t] \tag{4}$$

where

$$\boldsymbol{\rho}(\mathbf{r}_0, t)=\int_{t_0}^{t} ds\, \mathbf{u}[\mathbf{r}_0+\boldsymbol{\rho}(\mathbf{r}_0, s), s] \tag{5}$$

Note that $\boldsymbol{\rho}(\mathbf{r}_0, t)$ is precisely the Lagrangian trajectory of a fluid particle that is positioned at $\mathbf{r}_0$ at time $t=t_0$. The field $\mathbf{v}(\mathbf{r}_0, t_0 \mid \mathbf{r}, t)$ is simply the Eulerian field in the frame of reference of a single chosen material point $\boldsymbol{\rho}(\mathbf{r}_0, t)$. Next we define the field of velocity differences $\mathscr{W}(\mathbf{r}_0, t_0 \mid \mathbf{r}, \mathbf{r}', t)$:

$$\mathscr{W}(\mathbf{r}_0, t_0 \mid \mathbf{r}, \mathbf{r}', t) \equiv \mathbf{v}(\mathbf{r}_0, t_0 \mid \mathbf{r}, t)-\mathbf{v}(\mathbf{r}_0, t_0 \mid \mathbf{r}', t) \tag{6}$$

It was shown that the equation of motion of this field is independent of $t_0$, and we can omit this label altogether.

The fundamental statistical quantities in our study are the many-time, many-point, "fully-unfused," $n$-rank-tensor correlation function of the BL velocity differences $\mathscr{W}_j \equiv \mathscr{W}(\mathbf{r}_0 \mid \mathbf{r}_j, \mathbf{r}'_j, t_j)$. To simplify the notation we choose the following short hand notation:

$$X_j \equiv \{\mathbf{r}_j, \mathbf{r}'_j, t_j\}, \qquad x_j \equiv \{\mathbf{r}_j, t_j\}, \qquad \mathscr{W}_j \equiv \mathscr{W}(X_j) \tag{7}$$

$$\mathscr{F}_n(\mathbf{r}_0 \mid X_1 \cdots X_n) = \langle \mathscr{W}_1 \cdots \mathscr{W}_n \rangle \tag{8}$$

By the term "fully unfused" we mean that all the coordinates are distinct and all the separations between them lie in the inertial range. In particular the 2nd order correlation function written explicitly is

$$\begin{aligned} &\mathscr{F}_2^{\alpha\beta}(\mathbf{r}_0 \mid \mathbf{r}_1, \mathbf{r}'_1, t_1; \mathbf{r}_2, \mathbf{r}'_2, t_2) \\ &\quad = \langle \mathscr{W}^\alpha(\mathbf{r}_0 \mid \mathbf{r}_1, \mathbf{r}'_1, t_1)\, \mathscr{W}^\beta(\mathbf{r}_0 \mid \mathbf{r}_2, \mathbf{r}'_2, t_2) \rangle \end{aligned} \tag{9}$$

In addition to the $n$-order correlation functions the statistical theory calls for the introduction of a similar array of response or Green's functions. The most familiar is the 2nd order Green's function $G^{\alpha\beta}(\mathbf{r}_0 \mid X_1; x_2)$ defined by the functional derivative

$$G^{\alpha\beta}(\mathbf{r}_0 \mid X_1; x_2) = \left\langle \frac{\delta \mathscr{W}^\alpha(\mathbf{r}_0 \mid X_1)}{\delta \phi^\beta(\mathbf{r}_0 \mid x_2)} \right\rangle \tag{10}$$

In stationary turbulence these quantities depend on $t_1 - t_2$ only, and we can denote this time difference as $t$.

The simultaneous correlation function $\mathscr{T}_n$ is obtained from $\mathscr{F}_n$ when $t_1 = t_2 \cdots 3t_n$. In this limit one can use differences of Eulerian velocities, $\mathbf{w}(\mathbf{r}, \mathbf{r}', t) \equiv \mathbf{u}(\mathbf{r}', t) - \mathbf{u}(\mathbf{r}, t)$ instead of BL-differences; the result is the same. In statistically stationary turbulence the equal time correlation function is time independent, and we denote it as

$$\mathscr{T}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; ...; \mathbf{r}_n, \mathbf{r}'_n) = \langle \mathbf{w}(\mathbf{r}_1, \mathbf{r}'_1, t)\, \mathbf{w}(\mathbf{r}_2, \mathbf{r}'_2, t) \cdots \mathbf{w}(\mathbf{r}_n, \mathbf{r}'_n, t) \rangle \tag{11}$$

One expects that when all the separations $R_i \equiv |\mathbf{r}_i - \mathbf{r}'_i|$ are in the inertial range, $\eta \ll R_i \ll L$, the simultaneous correlation function is scale invariant in the sense that

$$\mathscr{T}_n(\lambda\mathbf{r}_1, \lambda\mathbf{r}'_1; \lambda\mathbf{r}_2, \lambda\mathbf{r}'_2; ...; \lambda\mathbf{r}_n, \lambda\mathbf{r}'_n) = \lambda^{\zeta_n} \mathscr{T}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; ...; \mathbf{r}_n, \mathbf{r}'_n) \tag{12}$$

While this statement appears plausible, it must stressed that this is an *assumption* which is fundamental to our theory. The homogeneous form implies the existence of the scaling exponents $\zeta_n$. Our second *assumption* is the widely accepted expectation that the scaling exponents $\zeta_n$ are universal in the sense that they are independent of the manner of turbulence excitation at least for some wide class of large-scale forcing.

The goal of our work is: *the calculation of the exponents $\zeta_n$ from first principles. This is the first aim of a statistical theory of turbulence.* All that is known a priori is that

$$\zeta_0 = 0, \qquad \zeta_3 = 1 \tag{13}$$

Unfortunately, a theory of same-time correlation functions cannot be closed upon itself. Unavoidably, different time correlation functions appear in the theory, and therefore it is necessary to know something about their scaling properties. It was proposed by many authors that the assumption of homogeneity (12) can be extended also to the different-time correlation functions $\mathscr{F}_n$, including their time arguments. This is however not the case, and in the context of turbulence, when the exponents $\zeta_n$ are anomalous, dynamical scaling is broken. The time correlation functions are "multiscaling" in the time variables. In ref. 9 it was shown that the scaling properties of the time correlation functions can be parametrized conveniently with the help of one scalar function $\mathscr{Z}(h)$. To understand this presentation, consider first the simultaneous function $\mathscr{T}_n(\mathbf{r}_1 \cdots \mathbf{r}'_n)$. Following the standard ideas of multi-fractals[2,6] the simultaneous function can be represented as

$$\mathscr{T}_n(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}'_n) = U^n \int_{h_{\min}}^{h_{\max}} d\mu(h) \left(\frac{R_n}{L}\right)^{nh + \mathscr{Z}(h)} \tilde{\mathscr{T}}_{n,h}(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1, \dots, \boldsymbol{\rho}'_n) \tag{14}$$

where $U$ is a typical velocity scale, and Greek coordinates stand for dimensionless (rescaled) coordinates, i.e.,

$$\boldsymbol{\rho}_j = \mathbf{r}_j / R_n, \qquad \boldsymbol{\rho}'_j = \mathbf{r}'_j / R_n \tag{15}$$

In Eq. (14) we have introduced the "typical scale of separation" of the set of coordinates

$$R_n^2 = \frac{1}{n} \sum_{j=1}^{n} |\mathbf{r}_j - \mathbf{r}'_j|^2 \tag{16}$$

At this point $L$ is an undetermined renormalization scale. At the end of the calculation we will find that it is the outer scale of turbulence. The function $\mathscr{Z}(h)$ is related to the scaling exponents $\zeta_n$ via the saddle point requirement[6]

$$\zeta_n = \min_h \left[ nh + \mathscr{Z}(h) \right] \tag{17}$$

This identification stems from the fact that the integral in (14) is computed in the limit $R_n/L \to 0$ via the steepest descent method. Neglecting logarithmic corrections one finds that $\mathscr{T}_n \propto R_n^{\zeta_n}$.

The representation (14) has appeared before in the theory of turbulence as a phenomenological idea where our function $\mathscr{Z}(h)$ was written as $\mathscr{Z}(h) = 3 - D(h)$, where $D(h)$ was understood as the fractal dimension of the set of spatial points that support a local Hölder exponent $h$. We do not necessarily accept this geometric interpretation. Nevertheless, in paper I, and again in Section 2 of this paper, we show that the representation (14) stems from an exact rescaling symmetry of the equations for the time correlation functions $\mathscr{F}_n$, and that the theory leads to a calculation of $\mathscr{Z}(h)$ from first principles, and therefore, through (17), to that of the scaling exponents $\zeta_n$. The original appearance of this form was based on the physical intuition that there are velocity field configurations that are characterized by different scaling exponents $h$. For different orders $n$ the main contribution comes from that value of $h$ that determines the position of the saddle point in the integral (14). It is convenient to introduce a dimensional quantity $\mathscr{T}_{n,h}$ according to

$$\mathscr{T}_{n,h}(\mathbf{r}_1, \mathbf{r}'_1,..., \mathbf{r}'_n) = U^n \left(\frac{R_n}{L}\right)^{nh+\mathscr{Z}(h)} \tilde{T}_{n,h}(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1,..., \boldsymbol{\rho}'_n) \tag{18}$$

Dimensional quantities of this type will play an important role in our theory. In particular their rescaling properties will be used to organize a $\mathscr{Z}$-covariant theory, see Section 3. This quantity re-scales like

$$\mathscr{T}_{n,h}(\lambda\mathbf{r}_1, \lambda\mathbf{r}'_1,..., \lambda\mathbf{r}'_n) = \lambda^{nh+\mathscr{Z}(h)} \mathscr{T}_{n,h}(\mathbf{r}_1, \mathbf{r}'_1,..., \mathbf{r}'_n) \tag{19}$$

Below we will use a shorthand notation for such rescaling laws, $\mathscr{T}_{n,h} \to \lambda^{nh+\mathscr{Z}(h)} \mathscr{T}_{n,h}$.

The intuition behind the representation (14) is extended to the time domain. The particular velocity configurations that are characterized by an exponent $h$ also display a typical time scale $t_{R,h}$ which is written as

$$t_{R,h} \sim \frac{R}{U}\left(\frac{L}{R}\right)^h \tag{20}$$

Accordingly we write a temporal multiscaling representation for the time dependent function

$$\mathscr{F}_n(\mathbf{r}_0 \mid X_1,..., X_n) = U^n \int_{h_{\min}}^{h_{\max}} d\mu(h) \left(\frac{R_n}{L}\right)^{nh+\mathscr{Z}(h)} \tilde{\mathscr{F}}_{n,h}(\mathbf{r}_0 \mid \Xi_1, \Xi_2,..., \Xi_n) \tag{21}$$

where $\tilde{\mathscr{F}}_{n,h}$ depends on the dimensionless (rescaled) space and time coordinates

$$\Xi_j \equiv (\boldsymbol{\rho}_j, \boldsymbol{\rho}'_j, \tau_j), \qquad \tau_j = t_j / t_{R_n, h} \tag{22}$$

As before, we introduce the dimensional quantity $\mathscr{F}_{n,h}$ according to:

$$\mathscr{F}_n(\mathbf{r}_0 \mid X_1, \dots, X_n) = U^n \left(\frac{R_n}{L}\right)^{nh + \mathscr{Z}(h)} \tilde{\mathscr{F}}_{n,h}(\mathbf{r}_0 \mid \Xi_1, \Xi_2, \dots, \Xi_n) \tag{23}$$

where the short notation $X_i$ is defined by Eq. (7). We require that the function $\tilde{\mathscr{F}}_{n,h}(\mathbf{r}_0 \mid \Xi_1, \dots, \Xi_n)$ will be identical to $\tilde{T}_{n,h}(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1, \dots, \boldsymbol{\rho}'_n)$ when its rescaled time arguments are all the same. We explain in Section 2 why Eq. (23) is *required* by the rescaling symmetries of the equations of motion for the time correlation functions, thus opening up a strategy of computing the scaling exponents by computing $\mathscr{Z}(h)$ first.

### A. Fundamental Equations on an "*h*-slice"

In paper I we showed that the $n$th order correlation functions satisfy, in the limit of infinite Reynolds number, an exact hierarchy of equations of the form:

$$\frac{\partial}{\partial t_1} \mathscr{F}_n(\mathbf{r}_0 \mid X_1, \dots, X_n) + \int d\tilde{\mathbf{r}}\, \boldsymbol{\gamma}(\mathbf{r}_1, \mathbf{r}'_1, \tilde{\mathbf{r}})\, \mathscr{F}_{n+1}(\mathbf{r}_0 \mid \tilde{X}, \tilde{X}, X_2, \dots, X_n) = 0 \tag{24}$$

where $X_j$ is defined by Eq. (7) and $\tilde{X}$ stands for

$$\tilde{X} = \{\tilde{\mathbf{r}}, \mathbf{r}_0, t_1\} \tag{25}$$

and the vertex function $\gamma^{\alpha\mu\sigma}(\mathbf{r}_1, \mathbf{r}'_1, \tilde{\mathbf{r}})$ is defined as

$$\begin{aligned} \gamma^{\alpha\mu\sigma}(\mathbf{r}_1, \mathbf{r}'_1, \tilde{\mathbf{r}}) = \frac{1}{2} \Big\{ & [P^{\alpha\mu}(\mathbf{r}_1 - \tilde{\mathbf{r}}) - P^{\alpha\mu}(\mathbf{r}'_1 - \tilde{\mathbf{r}})] \frac{\partial}{\partial \tilde{r}_\sigma} \\ & + [P^{\alpha\sigma}(\mathbf{r}_1 - \tilde{\mathbf{r}}) - P^{\alpha\sigma}(\mathbf{r}'_1 - \tilde{\mathbf{r}})] \frac{\partial}{\partial \tilde{r}_\mu} \Big\} \end{aligned} \tag{26}$$

The projection operator $P^{\alpha\beta}$ was defined explicitly in paper I.

In this section we introduce equations on an "$h$-slice" and consider their symmetry under rescaling. Firstly we discuss the equations for correlation functions, and secondly equations for Green's functions.

## B. Correlation Functions and the Rescaling Group

Examining Eqs. (24) one realizes that they are *linear* in the set of correlation functions $\mathscr{F}_n$ with $n = 2, 3,\ldots$. Accordingly, we can seek a solution in the form of a weighted sum of solutions, to be denoted as $\mathscr{F}_{n,h}$,

$$\int_{h_{\min}}^{h_{\max}} d\mu(h) \left\{ \frac{\partial}{\partial t_1} \mathscr{F}_{n,h}(\mathbf{r}_0 \mid X_1,\ldots, X_n) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}'_1, \tilde{\mathbf{r}})\, \mathscr{F}_{n+1,h}(\mathbf{r}_0 \mid \tilde{X}, \tilde{X}, X_2,\ldots, X_n) \right\} = 0 \tag{27}$$

with $\mu(h)$ some positive measure. For the sum

$$\int_{h_{\min}}^{h_{\max}} d\mu(h)\, \mathscr{F}_{n,h}$$

to be a solution we must satisfy separately the equations in curly parentheses. In other words, we will seek solutions to the equations

$$\frac{\partial}{\partial t_1} \mathscr{F}_{n,h}(\mathbf{r}_0 \mid X_1,\ldots, X_n) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}'_1, \tilde{\mathbf{r}})\, \mathscr{F}_{n+1,h}(\mathbf{r}_0 \mid \tilde{X}, \tilde{X}, X_2,\ldots, X_n) = 0 \tag{28}$$

We refer to these equations as the equations on an "$h$-slice." It is important to analyze their properties under rescaling. To this aim recall that the Euler equation is invariant to rescaling according to

$$\mathbf{r}_i \to \lambda \mathbf{r}_i, \qquad t_i \to \lambda^{1-h} t_i, \qquad \mathbf{u} \to \lambda^h \mathbf{u} \tag{29}$$

for any value of $\lambda$ and $h$. On the basis of this alone one could guess that Eqs. (28) are invariant under the rescaling group

$$\mathbf{r}_i \to \lambda \mathbf{r}_i, \qquad t_i \to \lambda^{1-h} t_i, \qquad \mathscr{F}_{n,h} \to \lambda^{nh} \mathscr{F}_{n,h} \tag{30}$$

It is easy to check that this is true. In fact we can see that Eqs. (28) are invariant to a broader rescaling group, i.e.,

$$\mathbf{r}_i \to \lambda \mathbf{r}_i, \qquad t_i \to \lambda^{1-h} t_i, \qquad \mathscr{F}_{n,h} \to \lambda^{nh + \mathscr{Z}(h)} \mathscr{F}_{n,h} \tag{31}$$

*with an n independent function* $\mathscr{Z}(h)$. This extra freedom is an exact result of the structure of the equations (28). We reiterate that the function $\mathscr{Z}(h) = 3 - D(h)$ appeared originally in the phenomenology of turbulence as an ad hoc model of multi-fractal turbulence. At this point it appears as a nontrivial and *exact* property of the chain of equations of the statistical theory of turbulence.

We will show below that the preservation of this rescaling symmetry will lead to a theory in which power counting plays no role, and in which new, non-Kolmogorov scaling solutions are possible. As a consequence of the lack of power counting, information about the scaling exponents $\zeta_n$ is obtainable only from the solvability conditions of these equations. In other words, the information is contained in coefficients rather than in power counting. The spatial derivative in the vertex on the RHS brings down the unknown function $\mathscr{X}(h)$, and its calculation will be an integral part of the computation of the exponents. It will serve the role of a generalized eigenvalue of the theory.

Of course, we cannot consider the hierarchy of equations (28) in its entirety. We need to find ways to close this equation, and this is the main subject of Section 3. The main idea in choosing an appropriate closure is to preserve the essential rescaling symmetry of the problem. We will argue below that there are many different possible closures, but most of them do not preserve this rescaling symmetry. We will introduce the notion of $\mathscr{X}$ covariance, and demand that the closure does not destroy the $h$-slice rescaling symmetry (31).

### C. Temporal Multiscaling in the Green's Functions

In addition to the $n$-order correlation functions the statistical theory calls for the introduction of a similar array of nonlinear response or Green's functions $\mathscr{G}_{m,n}$. These represent the response of the direst product of $m$ BL-velocity differences to $n$ perturbations. In particular

$$\mathscr{G}_{2,1}(\mathbf{r}_0 \mid X_1, X_2; x_3) = \left\langle \frac{\delta[\mathscr{W}(\mathbf{r}_0 \mid X_1)\, \mathscr{W}(\mathbf{r}_0 \mid X_2)]}{\delta\boldsymbol{\phi}(\mathbf{r}_0 \mid x_3)} \right\rangle$$

$$\mathscr{G}_{3,1}(\mathbf{r}_0 \mid X_1, X_2, X_3; x_4) = \left\langle \frac{\delta[\mathscr{W}(\mathbf{r}_0 \mid X_1)\, \mathscr{W}(\mathbf{r}_0 \mid X_2)\, \mathscr{W}(\mathbf{r}_0 \mid X_3)]}{\delta\boldsymbol{\phi}(\mathbf{r}_0 \mid x_4)} \right\rangle \tag{32}$$

Note that the Green's function $\mathbf{G}$ of Eq. (10) is $\mathscr{G}_{1,1}$ in this notation. The set of Green's functions $\mathscr{G}_{n,1}$ satisfies a hierarchy of equations that in the limit of infinite Reynolds number is written as

$$\begin{aligned}
&\frac{\partial}{\partial t_1} \mathscr{G}_{n,1}^{\alpha\beta\cdots\psi\omega}(\mathbf{r}_0 \mid X_1, X_2,\dots, X_n; x_{n+1}) \\
&\qquad + \int d\tilde{\mathbf{r}}\, \gamma^{\alpha\mu\sigma}(\mathbf{r}_1, \mathbf{r}_1', \tilde{\mathbf{r}})\, \mathscr{G}_{n+1,1}^{\mu\sigma\beta\gamma\cdots\psi\omega}(\mathbf{r}_0 \mid \tilde{X}_1, \tilde{X}_1, X_2,\dots, X_n; x_{n+1}) \\
&\quad = \mathscr{G}_{n,1}^{(0)\,\alpha\beta\cdots\omega}(\mathbf{r}_0 \mid X_1, X_2,\dots, X_n, \mathbf{r}_{n+1}, t_1+0)\, \delta(t_1 - t_{n+1})
\end{aligned}$$

The bare Green's function of $(n, 1)$ order on the RHS of this equation is given by the following decomposition:

$$\mathscr{G}^{(0)\alpha\beta\cdots\psi\omega}_{n,1}(\mathbf{r}_0 \mid X_1, X_2,..., X_n, \mathbf{r}_{n+1}, t_1+0)$$
$$\equiv \mathscr{G}^{(0)\alpha\omega}_{1,1}(\mathbf{r}_0 \mid \mathbf{r}_1, \mathbf{r}'_1, \mathbf{r}_{n+1}, +0)\, \mathscr{F}^{\beta\gamma\cdots\psi}_{n-1}(\mathbf{r}_0 \mid X_2,..., X_n) \tag{33}$$

Note that these functions serve as the initial conditions for Eqs. (33) at times $t_{n+1} = t_1$.

The form of these equations is very close to the hierarchy satisfied by the correlation functions, and it is advantageous to use a similar temporal multiscaling representation for the nonlinear Green's functions:

$$\mathscr{G}_{n,1}(\mathbf{r}_0 \mid X_1, X_2,..., X_n; x_{n+1})$$
$$= \int_{h_{\min}}^{h_{\max}} d\mu(h)\, \mathscr{G}_{n,1,h}(\mathbf{r}_0 \mid X_1, X_2,..., X_n; x_{n+1}) \tag{34}$$

From the rescaling symmetry of the Euler equation we could guess the rescaling properties $\mathscr{G}_{n,1,h} \to \lambda^{(n-1)h-3}\mathscr{G}_{n,1,h}$. As before, the equations support a broader rescaling symmetry group,

$$\mathbf{r} \to \lambda \mathbf{r}, \qquad t \to \lambda^{1-h} t, \qquad \mathscr{G}_{n,1,h} \to \lambda^{(n-1)h-3+\mathscr{X}_G(h)}\mathscr{G}_{n,1,h} \tag{35}$$

In fact the choice of the scalar function $\mathscr{X}_G(h)$ is constrained by the initial conditions on the Green's functions. From Eq. (33) we learn that the Green's functions depend on $\mathscr{X}(h)$ via the appearance of the correlation functions. This means that $\mathscr{X}_G(h)$ and $\mathscr{X}(h)$ are related. A simple calculation leads to the conclusion that

$$\mathscr{X}_G(h) = \mathscr{X}(h) \tag{36}$$

In this subsection we displayed explicitly only the hierarchy of equations satisfied by $\mathscr{G}_{n,1}$. In this hierarchy the first equation corresponds to $n = 1$, the second to $n = 2$, etc. Similar hierarchies with $n = 1, 2,...$ can be derived for any family of higher order Green's function $\mathscr{G}_{n,m}$ with $m = 2, 3,...$. After introducing the multiscaling representation, we can consider the corresponding Green's functions on an "$h$-slice," $\mathscr{G}_{n,m,h}$ and show that the equations they satisfy have the rescaling symmetry of the Euler equation with a $\mathscr{X}(h)$ freedom. In all these equations the initial value terms force the scalar function $\mathscr{X}(h)$ to be $m$-independent.

## 3. $\mathscr{L}$-COVARIANT CLOSURES

Faced with infinite chains of equations, one needs to truncate at a certain order. Simply by truncating one obtains a set of equations which is not closed upon itself. It is then customary to express the highest order quantities in the truncated set of equations in terms of lower order quantities. This turns the set of equation into a nonlinear set. Such an approach is not guaranteed to preserve the essential rescaling symmetries (31) and (35) of the equations. In this section we develop a systematic method to close the hierarchies of equations for correlation and Green's functions on an "$h$-slice" in a way that preserves the rescaling symmetry.

The lowest order closure involves five equations on an "$h$-slice." The first two belong to the $\mathscr{F}_n$ hierarchy:

$$\begin{aligned} &\frac{\partial}{\partial t_1}\mathscr{F}_{2,h}(X_1, X_2) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tilde{\mathbf{r}})\, \mathscr{F}_{3,h}(\tilde{X}, \tilde{X}, X_2) = 0 \\ &\frac{\partial}{\partial t_1}\mathscr{F}_{3,h}(X_1, X_2, X_3) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tilde{\mathbf{r}})\, \mathscr{F}_{4,h}(\tilde{X}, \tilde{X}, X_2, X_3) = 0 \end{aligned} \tag{37}$$

The next pair of equations belongs to the hierarchy of $\mathscr{G}_{n,1}$. Using the representation (34) in (33) we derive:

$$\begin{aligned} &\frac{\partial}{\partial t_1}\mathscr{G}_{1,1,h}(X_1; x_2) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tilde{\mathbf{r}})\, \mathscr{G}_{2,1,h}(\tilde{X}, \tilde{X}; x_2) \\ &\qquad = \mathbf{G}_h^{(0)}(X_1; x_2)\, \delta(t_1 - t_2) \\ &\frac{\partial}{\partial t_1}\mathscr{G}_{2,1,h}(X_1, X_2; x_3) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tilde{\mathbf{r}})\, \mathscr{G}_{3,1,h}(\tilde{X}, \tilde{X}, X_2; x_3) = 0 \end{aligned} \tag{38}$$

Here $\mathbf{G}_h^{(0)}$ stands for the bare Green's function on an "$h$-slice." The bare Green's function will be negligible compared to the "anomalous" Green's functions that we seek, and eventually Eq. (38) will be considered homogeneous. The fifth needed equation is the first equation from the hierarchy of Green's functions $\mathscr{G}_{n,2,h}$:

$$\frac{\partial}{\partial t_1}\mathscr{G}_{1,2,h}(X_1; x_2, x_3) + \int d\tilde{\mathbf{r}}\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tilde{\mathbf{r}})\, \mathscr{G}_{2,2,h}(\tilde{X}, \tilde{X}; x_2, x_3) = 0 \tag{39}$$

These five equations are presented symbolically in Fig. 2, and the basic symbols are defined in the Fig. 1. We now show that in the first step of the closure procedure these five equations can be considered as $\mathscr{L}(h)$-covariant

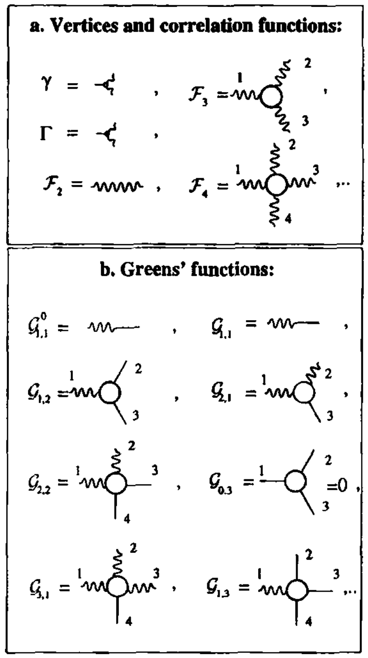


Fig. 1. The diagrammatic notation of the basic objects of the theory. Panel a: the vertex $\gamma$ and the correlation functions $\mathscr{F}_n$ with $n = 2, 3, 4$. Panel b: the bare Green's function $\mathscr{G}^0_{1,1}$ (thin line), and the dressed Green's functions $\mathscr{G}_{n,m}$. Objects with only straight tails are identically zero.

equations for five unknowns. These five objects are the 2nd order correlation function $\mathscr{F}_{2,h}$, the regular Green's function $\mathscr{G}_{1,1,h}$, and three types of triple vertices.

The vertices are introduced in Figs. 3 and 4. We have in these figures three relationships that *define* the vertices $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{C}_h$ *on an "h-slice"* in terms of $\mathscr{F}_{3,h}$, $\mathscr{G}_{2,1,h}$ and $\mathscr{G}_{1,2,h}$. Note that there is no notion of perturbation theory here—we simply define the three vertices in terms of objects that appear in Eqs. (37), (39) and (38).

Equations (37) involves the 4th point correlator $\mathscr{F}_{4,h}$, Eq. (38) and Eq. (39) involve $\mathscr{G}_{3,1,h}$ and $\mathscr{G}_{2,2,h}$. These are 4th order objects, and we present them in Figs. 5 in terms of all the possible decompositions made of lower order objects, and in addition new ("irreducible") contributions which *are defined* by these relations. In order to have a consistent definition we need to add to this game the Green's function $\mathscr{G}_{1,3,h}$. In the context of the 4th order objects the irreducible contributions are denoted symbolically as empty squares. There are four of them, and we denote them as $\mathbf{D}_{3,1,h}$, $\mathbf{D}_{2,2,h}$, $\mathbf{D}_{1,3,h}$ and $\mathbf{D}_{0,4,h}$. The first index stands for the number of wavy

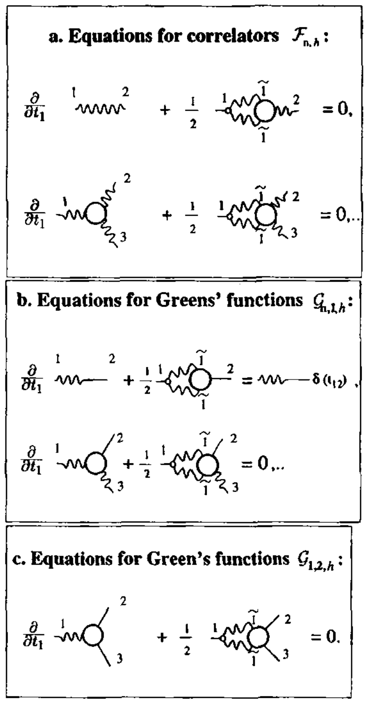


Fig. 2. The symbolic representation of the first equations of motion in the hierarchy of equations for correlation functions and Green's functions. A short wavy line stands for a velocity field, and a short straight line stands for the forcing. A long wavy line is the 2-point correlation function, and a short wavy line connected to a short straight line is the standard Green's function. A circle connecting $n$ wavy lines stands for an $n$th order correlation function, and a circle with $n$ wavy lines and $m$ straight lines stands for a Green's function with $n$ velocities and $m$ forcing. The Green's function represented by a thin line is the bare Green's function.

"tails" and the second index for the number of straight tails of the empty square.

## A. Systematic Closure

*The first step of closure consists in discarding the irreducible empty squares.* After doing so, we remain with precisely five equations (37)–(39) for five unknown functions. In the next step of closure we retain the empty squares as defined by their relations to the 4th order correlation and Green's function, and *add to the list of equations on an "h-slice" the equations of motion for the 4th order objects*, i.e., $\mathcal{F}_{4,h}$, $\mathcal{G}_{3,1,h}$, $\mathcal{G}_{2,2,h}$ and $\mathcal{G}_{1,3,h}$.

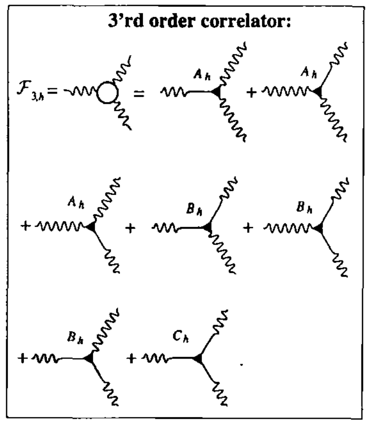


Fig. 3. The representation of the 3rd order correlation function on an "$h$-slice" in terms of 2-point quantities and three vertices. Note that this is *not* a perturbative expansion, but a definition of the vertices on an "$h$-slice." The definition of the vertices is completed in Fig. 4.

In total we have at this point nine equations. These equations will involve four 5th order objects, i.e., $\mathscr{F}_{5,h}$, $\mathscr{G}_{4,1,h}$, $\mathscr{G}_{3,2,h}$ and $\mathscr{G}_{2,3,h}$. Each of these new objects can be written now in terms of all the contributions that can be made from low order objects, plus irreducible 5th order vertices that we denote as empty pentagons. The second step of closure consists in discarding the empty pentagons. This gives us precisely nine equations for

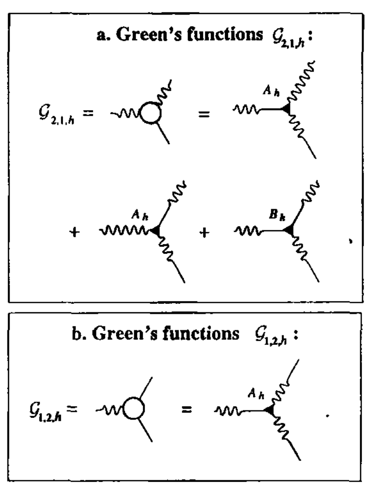


Fig. 4. This figure completes the definition of the three vertices in terms of the 3rd order correlation and Green's functions.

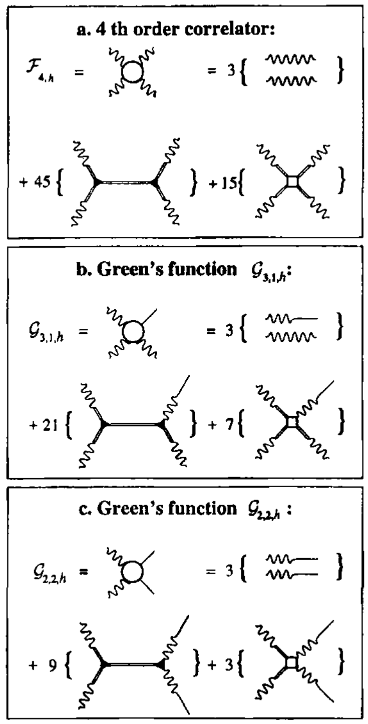


Fig. 5. Representation of the 4th order quantities in terms of lower order quantities, and additional "irreducible" 4th order vertices. Note again that this is not a perturbative expansion, but a definition of the 4th order vertices. A double line stands for "wavy or straight line."

nine unknown functions, i.e., $\mathcal{F}_{2,h}$, $\mathcal{G}_{1,1,h}$, $\mathbf{A}_h$, $\mathbf{B}_h$, $\mathbf{C}_h$, and the four empty square vertices $\mathbf{D}_{3,1,h}$, $\mathbf{D}_{2,2,h}$, $\mathbf{D}_{1,3,h}$ and $\mathbf{D}_{0,4,h}$.

The procedure is now clear in its entirety. At the $n$th step of the closure we will discard the $n+3$th irreducible contributions, and will have precisely the right number of equations on an "$h$-slice" to solve for the remaining unknowns. We should stress that this procedure is not perturbative since we solve the exact equations on an "$h$-slice." Our presentation of $n$th order objects on the "$h$-slice" in terms of lower order ones is also exact, it just *defines* at the $n$th step of the procedure a group of $(n+3)$th new vertices. These vertices are only solved for at the $(n+1)$th step of the procedure, when the $(n+4)$th vertices are discarded.

It would be only fair to say that the idea for this procedure came from a careful examination of the fully renormalized perturbation theory for this problem.[1] In that procedure one can *derive* equations for the $n$th order objects that appear symbolically like Fig. 5, etc. In addition, one can at each step of the procedure have an infinite expansion for the irreducible contributions. Nevertheless, the procedure explained above is different; firstly, it is consistently developed on an "$h$-slice," whereas the renormalized perturbation theory is done for the standard statistical objects. Secondly, at no point is there any infinite expansion whose convergence properties are dubious. We just go through a set of explicit definitions solving an exact set of equations. The only question that needs to be understood is the speed of convergence of this scheme in terms of the scalar function $\mathscr{Z}(h)$ which parametrizes the anomalous behavior.

Lastly, we should explain that the closure procedure employed here leaves no free parameters for later use. In particular, by discarding at every step the irreducible contribution of the $n$-pentagon, we may lose the exact conditions (13). We may construct another closure procedure in which the $n$-pentagon is related, with the help of free parameters, to lower order objects, and fit the free parameter at a later stage to agree with (13). For the sake of the present demonstration we will stick to the leaner procedure in which no free parameters are produced at this stage. We will see in the sequel that the approximate procedure to solve the lowest order closure will introduce two free parameters anyway, and these will suffice to satisfy (13).

## B. $\mathscr{Z}$-covariance

A crucial property of our closure procedure is that it guarantees that power counting remains irrelevant on an "$h$-slice" for an arbitrary step of the procedure, and that the scalar function $\mathscr{Z}(h)$ cannot be computed from power counting. To see this we need to find the rescaling properties of the triple and higher order vertices. We start with the triple vertices $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{C}_h$. The first one is defined by its relation to $\mathscr{G}_{1,2,h}$, see Fig. 4b. Using the facts that

$$\mathscr{F}_{2,h} \to \lambda^{2h+\mathscr{Z}(h)}\mathscr{F}_{2,h} \tag{40}$$

$$\mathscr{G}_{1,1,h} \to \lambda^{\mathscr{Z}(h)-3}\mathscr{G}_{1,1,h} \tag{41}$$

$$\mathscr{G}_{1,2,h} \to \lambda^{-h+\mathscr{Z}(h)-6}\mathscr{G}_{1,2,h} \tag{42}$$

we find that $\mathbf{A}_h$ has to transform according to

$$\mathbf{A}_h \to \lambda^{2h-2\mathscr{Z}(h)-9}\mathbf{A}_h \tag{43}$$

Armed with this knowledge we proceed to the definition of $\mathbf{B}_h$ through its relation to $\mathscr{G}_{2,1,h}$, equation in Fig. 4a. We can check that the rescaling property of

$$\mathscr{G}_{2,1,h} \to \lambda^{h+\mathscr{Z}(h)-3}\mathscr{G}_{2,1,h} \tag{44}$$

agrees exactly with the two terms that contain the vertex $\mathbf{A}_h$ on the RHS. Accordingly also the term containing $\mathbf{B}_h$ has to transform in the same way, leading to

$$\mathbf{B}_h \to \lambda^{4h-2\mathscr{Z}(h)-6}\mathbf{B}_h \tag{45}$$

In making this assertion we assumed that there is no cancellation of the leading terms in the equation. Otherwise the vertex $\mathbf{B}_h$ would be smaller. Lastly we use the definition of $\mathbf{C}_h$ by the relation to $\mathscr{F}_{3,h}$, see Fig. 3, that transforms like

$$\mathscr{F}_{3,h} \to \lambda^{3h+\mathscr{Z}(h)}\mathscr{F}_{3,h} \tag{46}$$

All the terms that include $\mathbf{A}_h$ and $\mathbf{B}_h$ have the same rescaling exponent as that of $\mathscr{F}_{3,h}$. We can therefore find the rescaling exponent of $\mathbf{C}_h$:

$$\mathbf{C}_h \to \lambda^{6h-2\mathscr{Z}(h)-3}\mathbf{C}_h \tag{47}$$

We again assumed that there is no cancellation of the leading terms in the equation for $\mathscr{F}_{3,h}$. If there is cancellation, the vertex $\mathbf{C}_h$ can be smaller.

At this point we can check that the first step in our closure scheme leads to a $\mathscr{Z}(h)$ covariant procedure. Consider firstly the three contributions of Gaussian decomposition which are first on the RHS of the equation in Fig. 5a. These rescale like $\lambda^{4h+2\mathscr{Z}(h)}$, and their ratio to the LHS is proportional to $(R/L)^{\mathscr{Z}(h)}$. For $\mathscr{Z}(h)$ positive these contributions become irrelevant in the limit $(R/L) \to 0$. The 45 contributions that come next contain pairs of triple vertices, and we need to use the rescaling properties (43), (45), and (47) to find their rescaling exponents. We find that they *all share the same rescaling exponent*. In hindsight this should not be surprising. This is a result of the *assumption* that in the definitions of the three vertices there are no cancellations in the leading scaling behavior. Thus the rescaling exponent of all the 45 contributions could be obtained from analyzing one of them. The nontrivial fact is that the common rescaling of all these terms is exactly the rescaling of the LHS of the equation, which is $\lambda^{4h+\mathscr{Z}(h)}$. This means that our closure for the 4th order correlation functions cannot introduce power counting. Note that the rescaling neutrality with respect to counting of $h$ and of natural numbers follows from the rescaling symmetry of the Euler equation, and is shared also by Gaussian contributions. On the other hand the neutrality with respect to $\mathscr{Z}(h)$ is nontrivial, and follows from a judicious choice of the proposed closure scheme. We will refer this property as $\mathscr{Z}$-covariance.

It is important to understand now that the proposed closures for the other 4th order objects, like Fig. 5b are also $\mathscr{Z}$-covariant. All that changes is the number of wavy and straight tails on the LHS and RHS of the equations, and the rescaling exponents change in the same way on the two sides of the equation.

We can now consider the next step of closure, taking into account the irreducible 4th order vertices (empty squares), discarding the 5th order empty pentagons. The procedure follows verbatim the one described above for the triple vertices and the rescaling exponents for the irreducible 4th order vertices $\mathbf{D}_{m,n,h}$ are determined:

$$\begin{aligned} \mathbf{D}_{m,n,h} &\to \lambda^{d_{m,n}(h)} \mathbf{D}_{m,n,h} \\ d_{m,n}(h) &= 2nh - 3\mathscr{Z}(h) - 3m - 4 \end{aligned} \tag{48}$$

We can check the terms that appear in the 5th order correlation and Green's functions, which are made of combinations of triple and 4th order vertices. We discover that all these terms share the same rescaling exponent, and that it agrees precisely with the rescaling of the 5th order correlation and Green's function. Accordingly also the second step of closure is $\mathscr{Z}$-covariant.

It becomes evident that we develop a systematic $\mathscr{Z}$-covariant closure scheme, and that power counting will not creep in at any step of the procedure. In the next section we show that the scalar function $\mathscr{Z}(h)$ can be computed from this scheme as a generalized eigenvalue. We also explain the role of the boundary conditions in the space of scales, and how the renormalization scale is chosen.

## 4. THE SCALAR FUNCTION $\mathscr{Z}$ AS A GENERALIZED EIGENVALUE

In this section we demonstrate that at any step of our closure, the scalar function $\mathscr{Z}(h)$ can be found only from a solvability condition. We will also explain the role of the boundary conditions in the space of scales in determining the scaling functions in this theory.

The point is really rather simple. First observe that our initial equations for correlation and Green's functions on an "$h$-slice," like (28) or (38)–(39) are *linear* functional equations. The equations for the correlation functions are not only linear, but also homogeneous. The equations for the Green's functions are not all homogeneous, but as we explained in the last section the inhomogeneous terms are much smaller than the homogeneous terms in the limit $(R/L) \to 0$, and they can be discarded. This means, of course, that all the functions on an "$h$-slice" can be determined only up to an overall numerical constant. At this point we may have even more than one overall free constant; on the face of it the equations for the correlation functions are independent of the equations for the Green's functions, and we can have different overall constants in the correlation and the Green's functions.

Every step of closure turns a set of linear functional equations into a set of nonlinear functional equations. We claim that nevertheless all the functions appearing in these equations can be determined only up to an overall numerical constant. The extra freedom of many possible constants disappears now, since the correlation and the Green's functions are coupled after closure. But one overall constant remains free. The reason is of course the property of $\mathscr{Z}$-covariance which leaves us with an over-all free constant $\lambda^{\mathscr{Z}(h)}$.

If our equations were linear, this freedom would have meant that we need to require the standard solvability condition that the linear operator had an eigenvalue zero, and the functions that we seek would have been identified as the "zero modes" associated with this eigenvalue. Our equations are not linear, and the solvability condition is not that simple. Nevertheless, even though we have nonlinear equations at every step of the closure procedure these equations are over-determined and have non-zero solutions only for particular numerical values of $\mathscr{Z}(h)$. We will demonstrate in the next section how this solvability condition comes about in the lowest order closure.

Even after determining $\mathscr{Z}(h)$ the statistical functions will be determined only up to a factor which may depend on $h$, which is the measure $\mu(h)$ in Eqs. (21) and (34). In order to determine this factor we will need to fit all correlation functions $\mathscr{T}_n$ to the boundary conditions in the space of scales. At that point the computed values of $\mathscr{Z}(h)$ will determine whether it is the inner or the outer scale of turbulence that appears as the renormalization scale. We will show in the lowest step of closure that the outer scale $L$ is selected.

## 5. LOWEST STEP OF CLOSURE

In the rest of this paper we demonstrate that the approach discussed above results in anomalous scaling. We will analyze the simplest possible situation, which results from the first step of closure where in addition we neglect (for simplicity) the triple vertices $\mathbf{B}_h$ and $\mathbf{C}_h$. We are thus not particularly interested at this stage of development in accurate calculations of the scaling exponents, but rather in a demonstration that the scalar function $\mathscr{Z}(h)$ can be obtained from a solvability condition. In particular we want to stress the point that exponents other than K41 can be obtained in this scheme.

When the vertices $\mathbf{B}_h$ and $\mathbf{C}_h$ are neglected, we need to focus on the equation that defines the vertex $\mathbf{A}_h$ through its relation to $\mathscr{G}_{1,2,h}$, Fig. 4b. Next consider the equations of motion of $\mathscr{F}_{2,h}$ and $\mathscr{G}_{1,2,h}$ which are shown in Figs. 2a and c. These equations involve two triple objects, $\mathscr{F}_{3,h}$ and $\mathscr{G}_{1,2,h}$ and one quartic object, $\mathscr{G}_{2,2,h}$. These objects are represented in terms of the vertex $\mathbf{A}_h$ as shown in Fig. 6. Substituting these representation into the two equations of motion we find the equations shown in Figs. 7a and b. Examining the equation in panel b we observe that every diagram ends with a pair of Green's function $\mathscr{G}_{1,1,h}$. Accordingly the equation can be simplified by multiplying the equation from the right by two inverse

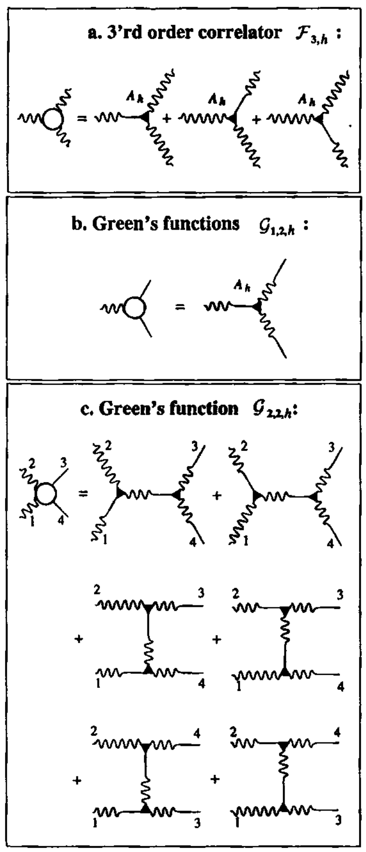


Fig. 6. Approximate presentation of the 3-point and 4-point objects on the $h$-slice in terms of the vertex $A_h$ only. In panel a all contributions involving $B_h$ and $C_h$ are neglected. Panel b is exact, and panel c follows from Fig. 5c neglecting the Gaussian contributions, the irreducible terms, and all the terms that involve $B_h$ and $C_h$ in the skeleton diagrams. These presentations are used to derive the equations in Fig. 7.

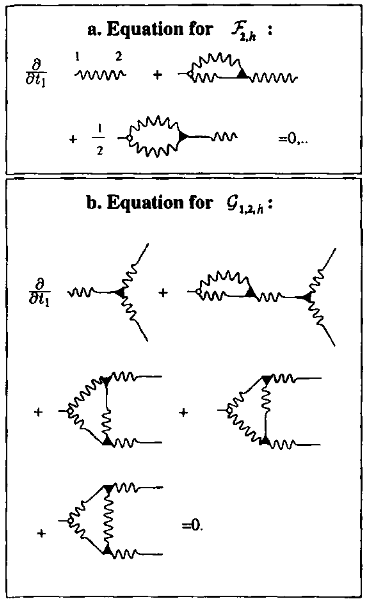


Fig. 7. Panel a: the equation of motion of $\mathscr{F}_{2,h}$ resulting from discarding vertices B and C. Panel b: The equation of motion of $\mathscr{G}_{1,2,h}$ resulting from discarding B and C and the irreducible contribution to 4th order vertices.

Green's functions $\mathscr{G}_{1,1,h}^{-1}$. After this step we observe that the Green's function and the vertex $\mathbf{A}_h$ always appear in a unique combination, which we will refer to as the "triplex" $T_h^{\alpha\beta\gamma}(\mathbf{r}_0 \mid X_1, X_2, X_3)$:

$$T_h^{\alpha\beta\gamma}(\mathbf{r}_0 \mid X_1, X_2, X_3) \equiv \int dx_4\, \mathscr{G}_{1,1,h}^{\alpha\delta}(\mathbf{r}_0 \mid X_1, x_4)\, A_h^{\delta\beta\gamma}(\mathbf{r}_0 \mid x_4, X_2, X_3) \tag{49}$$

The rescaling properties of this object can be read directly from this definition:

$$\mathbf{T}_h \to \lambda^{h-\mathscr{X}(h)-8}\mathbf{T}_h \tag{50}$$

We can extract from the lst equation in Fig. 7 an equation for the triplex, which together with the equation for $\mathscr{F}_{2,h}$ give us the final equations for our simplified closure. These final equations are shown in Fig. 8. We write these symbolic equations in analytic form for the dimensionless objects $\tilde{\mathbf{T}}_h$ and $\tilde{\mathscr{F}}_{2,h}$. We introduce the following three short hand notations:

$$F^{\alpha\beta}(12) \equiv \tilde{\mathscr{F}}^{\alpha\beta}_{2,h}(\mathbf{r}_0 \mid \Xi_1, \Xi_2) \tag{51}$$

$$T^{\alpha\beta\gamma}(123) \equiv \tilde{T}^{\alpha\beta\gamma}_{h}(\mathbf{r}_0 \mid \Xi_1, \Xi_2, \Xi_3) \tag{52}$$

$$\gamma(12)^{\alpha\beta\gamma} \equiv \gamma^{\alpha\beta\gamma}(\boldsymbol{\rho}_1, \boldsymbol{\rho}'_1, \boldsymbol{\rho}_2) \tag{53}$$

where dimensionless arguments $\Xi_i$ and $\boldsymbol{\rho}_i$ are determined by Eqs. (22) and (15). The two equations read:

$$\frac{\partial}{d\tau_1} F^{\alpha\beta}(12) + \left(\frac{R_3}{R_2}\right)^{3h + \mathscr{X}(h)} \int d3\, d4\, d5\, \gamma^{\alpha\delta\gamma}(13)\, \delta(\tau_1 - \tau_3) \times \left\{ T^{\gamma\sigma\rho}(345)\, F^{\delta\sigma}(34)\, F^{\beta\rho}(25) + \frac{1}{2} T^{\beta\sigma\delta}(245)\, F^{\gamma\sigma}(34)\, F^{\delta\rho}(35) \right\} = 0 \tag{54}$$

$$\frac{\partial}{d\tau_1} T^{\alpha\beta\gamma}(123) + \left(\frac{R_4}{R_3}\right)^{2h - \mathscr{X}(h) - 9} \int d4\, d5\, d6\, \gamma^{\alpha\delta\sigma}(14)\, \delta(\tau_1 - \tau_4) \times \{ T^{\sigma\mu\nu}(456)\, T^{\nu\beta\gamma}(623)\, F^{\delta\mu}(45) + T^{\sigma\gamma\mu}(435)\, T^{\mu\beta\nu}(526)\, F^{\delta\nu}(46) + T^{\delta\beta\mu}(425)\, T^{\mu\gamma\nu}(536)\, F^{\sigma\nu}(46) + T^{\sigma\gamma\nu}(436)\, T^{\delta\beta\mu}(425)\, F^{\mu\nu}(56) \} = 0 \tag{55}$$

These are two functional equations for two objects. We demonstrate now explicitly the freedom of rescaling by a constant factor: If we re-scale $F$ to $kF$, and $T$ to $T/k$, the equations remain invariant, and we cannot determine one constant. It means that for any $h$ there is a nonzero solution only for particular values of $\mathscr{X}(h)$.

## 6. REDUCTION TO A 3-SCALE SCALAR MODEL

The exact solution of the set of equations (54) and (55) still requires heavy analytic and numerical machinery due to their nonlinearity and the tensorial structure of the objects which depend on many space-time variables. This work is underway. In this paper we wish to simplify these equations as much as possible to demonstrate how anomalous scaling appears via solvability conditions in this problem. To this aim we will disregard the tensorial structure (to construct a scalar model), and consider the scalar objects as functions of fewer separations. Finally we will reduce the space integrations to a finite sum over what we believe are the most important contributions. The time integrations need to be respected much more carefully due to the breaking of dynamical scaling. To motivate this approach consider Eq. (37) in the situation when all the separations in the 2-point correlation function $\mathscr{F}_{2,h}$ are of the same order of magnitude $R$. We replace this function with a correlation function $F_{2,h}(R, t)$ which depends on one scalar separation and one time difference. The same equation contains $\mathscr{F}_{3,h}$, and we replace it with a scalar function $F_{3,h}(R_1, R_2, R_3, t_1, t_2, t_3)$ which is a function of only three scalar separations and three associated times. We choose $R_i$ in a scale invariant form

$$\frac{R_1}{R_2} = \frac{R_2}{R_3} = q \tag{56}$$

The motivation of this choice comes from the role of the integral over $\mathscr{F}_{3,h}$ in redistributing the energy among the scales. We expect the parameter $q$ to be of the order of 10[10, 11] due to the local nature of hydrodynamic interactions. But at this point $q$ is a free parameter that characterizes the range of scales from which the main contribution of the integrals comes. Due to stationarity our function depends on two time differences only. We wrote it as a function of three times for the sake of symmetry. Having in mind a function $F_{3,h}$ that depends symmetrically on three scales only, we can model the space integral in Eq. (37) as a sum of three contributions coming from scales larger than, of the order of, and smaller than $R$ (i.e., $R_2 = qR$, $R$ and $R/q$ respectively):

$$\frac{\partial F_{2,h}(R,t)}{\partial t} + \frac{a}{qR} F_{3,h}(q^2R, qR, R, 0, t, t) + \frac{b}{R} F_{3,h}\left(qR, R, \frac{R}{q}, t, 0, t\right) + \frac{cq}{R} F_{3,h}\left(R, \frac{R}{q}, \frac{R}{q^2}, t, t, 0\right) \approx 0 \tag{57}$$

The coefficients $a$, $b$, $c$ are the evaluation of the corresponding regions of integration and are of the order of unity, and they are modified by $1/Rq$, $1/R$ and $q/R$ correspondingly due to the evaluation of the vertex in the appropriate region.

To proceed we want to express the 3rd order correlation function $F_{3,h}$ in terms of lower order objects. To this aim examine Fig. 6, panel a. We see that $F_{3,h}$ can be expressed via one triplex and two 2nd order correlation functions. Of course, the vertex $A_h$ involves an integration over all separations, but in light of the approximation used to derive Eq. (57) we will employ only the "local" contributions to the 2nd order correlations, and write:

$$F_{3,h}(R_1, R_2, R_3, t_1, t_2, t_3) \approx \int_0^\infty dt'\, dt'' \{ T_h(R_1, R_2, R_3, t', t'') \times F_{2,h}(R_2, t_1 - t_2 + t')\, F_{2,h}(R_3, t_1 - t_3 + t'') + \text{term } 1 \to 2 \to 3 \to 1 + \text{term } 1 \to 3 \to 2 \to 1 \} \tag{58}$$

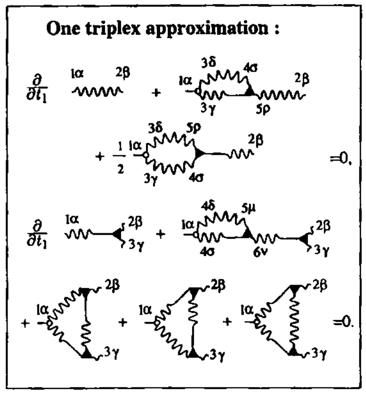


Fig. 8. The equations of motion in the one-triplex approximation. These equations are equivalent to those exhibited in Fig. 7; the second equation here follows simply from multiplying the second equation in Fig. 7 on the right by two inverse Green's functions.

In this equation we introduced the notation $T_h(R_1, R_2, R_3, t_1-t_2, t_1-t_3)$ to denote the scalar version of the triplex $\mathbf{T}_h$ that depends on three scalar separations and their associated times. We choose the first argument to be associated with the velocity coordinate in the Green's function. Therefore we need to require $t_2<t_1$, $t_3<t_1$ to have a finite contribution due to causality. Note that contrary to $F_{3,h}$, $T_h$ is not a symmetric function of $R_1, R_2, R_3$.

Substituting Eq. (58) in Eq. (57) results in one equation in terms of $F_{2,h}$ and $T_h$. The second equation is obtained from the second equation in Fig. 8. We choose $R_1=R$, $R_2=qR$ and $R_3=R/q$. With the assumed symmetry of $T_h$ we derive the equation

$$\frac{\partial}{\partial t_1} T_h\left(R, qR, \frac{R}{q}, t_1-t_2, t_1-t_3\right)+C_h(R, q, t_1-t_2, t_1-t_3)$$
$$+D_h(R, q, t_1-t_2, t_1-t_3)+E_h(R, q, t_1-t_2, t_1-t_3)=0 \qquad (59)$$

where the functions $C_h$, $D_h$ and $E_h$ represent the diagrams on the RHS of the equation in Fig. 8. To write them explicitly we employ the same 3-scale representation of the spatial integrals, ending up with

$$C_h(R, q, t_a, t_b)=\int_0^{\min\{t_a, t_b\}} dt\, \Sigma_h(R, q, t)\, T_h\left(R, qR, \frac{R}{q}, t_a-t, t_b-t\right) \qquad (60)$$

$$\Sigma_h(R, q, t) = \int_0^\infty dt' \left\{ \frac{aq}{R} \left[ Th\left(\frac{R}{q}, R, \frac{R}{q^2}, t, t'\right) F_2\left(\frac{R}{q^2}, t'\right) + T_h\left(\frac{R}{q^2}, \frac{R}{q}, R, t, t'\right) F_2\left(\frac{R}{q}, t'\right)\right] + \frac{b}{R}\left[ T_h\left(qR, R, \frac{R}{q}, t, t'\right) F_2\left(\frac{R}{q}, t'\right) + T_h\left(\frac{R}{q}, R, \frac{R}{q^2}, t', t\right) F_2(qR, t')\right] + \frac{c}{qR}\left[ T_h(q^2R, qR, R, t, t') F_2(qR, t') + T_h(qR, q^2R, R, t, t') F_2(q^2R, t')\right]\right\} \tag{61}$$

$$D_h(R, q, t_a, t_b) = \int_0^\infty dt\, dt' \left[ \frac{aq}{R} T_h\left(\frac{R}{q^2}, \frac{R}{q}, R, t, t_b\right) T_h\left(\frac{R}{q}, R, qR, t_a, t\right) F_2(R, t'-t) + \frac{c}{Rq}\left[ T_h\left(qR, R, \frac{R}{q}, t, t_a\right) T_h(q^2R, qR, R, t_a, t') F_2(R, t'-t)\right] \tag{62}$$

$$E_h(R, q, t_a, t_b) = \int_0^\infty dt\, dt' \left\{ \frac{aq}{R}\left[ T_h\left(\frac{R}{q}, R, qR, t_a, t'\right) \times T_h\left(R, \frac{R}{q}, \frac{R}{q^2}, t_b - t', t - t'\right) F_2\left(\frac{R}{q^2}, t\right) + T_h\left(\frac{R}{q^2}, \frac{R}{q}, R, t', t_b\right) T_h\left(R, qR, \frac{R}{q}, t_a - t', t - t'\right) F_2\left(\frac{R}{q}, t\right)\right] + \frac{c}{qR}\left[ T_h(q^2R, qR, R, t_a, t') T_h\left(R, qR, \frac{R}{q}, t - t', t_b - t'\right) F_2(qR, t) + T_h\left(qR, R, \frac{R}{q}, t', t_b\right) T(q^2R, qR, R, t - t', t_a - t') F_2(q^2R, t)\right]\right\} \tag{63}$$

To obtain Eqs. (60)–(63) we considered every integral over scales in the three domains of length scales according to Eq. (56). We note that the 2nd order correlation is considered as a function of one scale (diagonal in $R_1$, $R_2$), whereas the 3rd order correlation and the triplex connect three consecutive scales differing by a $q$ factor.

Since all our objects are defined in an "$h$-slice," they are all scale invariant, and we can make use of this fact to simplify the equations enormously. Consider the following scale invariant representation:

$$F_{2,h}(R,t) = U^2 \left(\frac{R}{L}\right)^{2h+\mathscr{Z}(h)} \tilde{F}_{2,h}\left(\frac{t}{t_{R,h}}\right) \tag{64}$$

$$F_{3,h}\left(qR, R, \frac{R}{q}, t_1, t_2, t_3\right) = U^3 \left(\frac{R}{L}\right)^{3h+\mathscr{Z}(h)} \tilde{F}_{3,h}\left(\frac{t_1}{t_{R,h}}, \frac{t_2}{t_{R,h}}, \frac{t_2}{t_{R,h}}\right) \tag{65}$$

These definitions follow verbatim from the general scheme described above. The scale invariant form of the triplex leaves some freedom of definition since the triplex is not a symmetric function of its arguments. We choose the following representation:

$$T_h\left(R, qR, \frac{R}{q}, t_a, t_b\right) = \frac{U}{L^2}\left(\frac{R}{L}\right)^{h-2-\mathscr{Z}(h)} \tilde{T}_{0,h}\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{66}$$

$$T_h\left(qR, R, \frac{R}{q}, t_a, t_b\right) = \frac{U}{L^2}\left(\frac{R}{L}\right)^{h-2-\mathscr{Z}(h)} q^{2h+\mathscr{Z}(h)} \tilde{T}_{-1,h}\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{67}$$

$$T_h\left(\frac{R}{q}, qR, R, t_a, t_b\right) = \frac{U}{L^2}\left(\frac{R}{L}\right)^{h-2-\mathscr{Z}(h)} q^{-2h-\mathscr{Z}(h)} \tilde{T}_{1,h}\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{68}$$

In writing these representations we extracted explicitly the power of $q$ in the last two definitions. This is not dictated by dimensional reasoning, and the factor could be absorbed in the rescaled quantities $\tilde{T}$. We found that this explicit notation leads to more compact forms of the resulting equations. Note that we distinguish three different objects $\tilde{T}$ according to which argument comes first, i.e., $R$, $qR$ or $R/q$. This stems from the fundamental fact that the triplex is not a symmetric function of its arguments.

In addition to the statistical objects we need to write a scale-invariant representation of the other objects discussed above, i.e., $\Sigma_h$, $C_h$, $D_h$ and $E_h$:

$$\Sigma_h(R,t) = t_{R,h}^{-2} \tilde{\Sigma}_h\left(\frac{t}{t_{R,h}}\right) \tag{69}$$

$$C_h(R, q, t_a, t_b) = \frac{U^2}{L}\left(\frac{R}{L}\right)^{2h-3-\mathscr{Z}(h)} \tilde{C}_h\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{70}$$

$$D_h(R, q, t_a, t_b) = \frac{U^2}{L}\left(\frac{R}{L}\right)^{2h-3-\mathscr{Z}(h)} \tilde{D}_h\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{71}$$

$$E_h(R, q, t_a, t_b) = \frac{U^2}{L}\left(\frac{R}{L}\right)^{2h-3-\mathscr{Z}(h)} \tilde{E}_h\left(\frac{t_a}{t_{R,h}}, \frac{t_b}{t_{R,h}}\right) \tag{72}$$

After substituting Eqs. (64)–(72) into Eq. (57) we find the dimensionless equation

$$\frac{d}{d\tau}\tilde{F}_{2,h}(\tau)+a\Lambda\tilde{F}_{3,h}(0,\lambda\tau,\lambda\tau)+b\tilde{F}_{3,h}(\tau,0,\tau)+\frac{c}{\Lambda}\tilde{F}_{3,h}\left(\frac{\tau}{\lambda},\frac{\tau}{\lambda},0\right)=0 \tag{73}$$

where we introduced the dimensionless time $\tau\equiv t/T_{R,h}$ and the factors

$$\lambda\equiv q^{1-h},\qquad \Lambda\equiv q^{1-3h-\mathscr{Z}(h)} \tag{74}$$

The same process applied to Eq. (58) results in a dimensionless equation for $\tilde{F}_3$:

$$\begin{aligned}\tilde{F}_{3,h}&(\tau_-,\tau_0,\tau_+)\\ &=\int_0^\infty d\tau'\,d\tau''[\tilde{T}_{-1,h}(\tau',\tau'')\,\tilde{F}_{2,h}(\tau_0-\tau_-+\tau')\,\tilde{F}_{2,h}(\lambda\tau_+-\lambda\tau_-+\lambda\tau'')\\ &\quad+\tilde{T}_{0,h}(\tau',\tau'')\,\tilde{F}_{2,h}(\tau_-/\lambda-\tau_0/\lambda+\tau'/\lambda)\,\tilde{F}_{2,h}(\lambda\tau_+-\lambda\tau_0+\lambda\tau'')\\ &\quad+\tilde{T}_{1,h}(\tau',\tau'')\,\tilde{F}_{2,h}(\tau_0-\tau_++\tau'')\,\tilde{F}_{2,h}(\tau_-/\lambda-\tau_+/\lambda+\tau'/\lambda)]\end{aligned} \tag{75}$$

In this equation the dimensionless times $\tau_-$, $\tau_0$ and $\tau_+$ are associated with the scales $qR$, $R$ and $R/q$ respectively.

The same process applied to Eq. (59) results in the dimensionless equation

$$\begin{aligned}\frac{d}{d\tau_1}&\tilde{T}_{0,h}(\tau_1-\tau_2,\tau_1-\tau_3)\\ &=\tilde{C}_h(\tau_1-\tau_2,\tau_1-\tau_3)+\tilde{D}_h(\tau_1-\tau_2,\tau_1-\tau_3)+\tilde{E}_h(\tau_1-\tau_2,\tau_1-\tau_3)\end{aligned} \tag{76}$$

The expressions for the dimensionless functions $\tilde{C}_h$, $\tilde{D}_h$ and $\tilde{E}_h$ follow from their definitions Eqs. (60)–(63). Explicitly,

$$\tilde{C}_h(\tau_a,\tau_b)=\int_0^{\min\{\tau_a,\tau_b\}}d\tau\,\tilde{\Sigma}(\tau)\,\tilde{T}_{0,h}(\tau_a-\tau,\tau_b-\tau) \tag{77}$$

In this equation $\Sigma(\tau)$ is given by

$$\begin{aligned}\tilde{\Sigma}_h(\tau)=\int_0^\infty d\tau'\{&(a\Lambda)[\tilde{F}_{2,h}(\tau')\,\tilde{T}_{0,h}(\lambda\tau,\tau'/\lambda)+\tilde{F}_{2,h}(\tau'/\lambda)\,\tilde{T}_{1,h}(\lambda\tau,\tau'/\lambda)]\\ &+b[\tilde{F}_{2,h}(\lambda\tau')\,\tilde{T}_{-1,h}(\tau,\tau')+\tilde{F}_{2,h}(\tau'/\lambda)\,\tilde{T}_{1,h}(\tau',\tau)]\\ &+(c/\Lambda)[\tilde{F}_{2,h}(\lambda\tau')\,\tilde{T}_{-1,h}(\lambda\tau',\tau/\lambda)+\tilde{F}_{2,h}(\tau')\,\tilde{T}_{0,h}(\lambda\tau',\tau/\lambda)]\}\end{aligned} \tag{78}$$

Next the expression for $\tilde{D}_h$ is

$$\tilde{D}_h(\tau_a, \tau_b) = \int_0^{\infty} d\tau\, d\tau' \{(a\Lambda)\, \tilde{T}_{1,h}(\tau, \lambda\tau_b)\, \tilde{T}_{1,h}(\tau_a, \tau'/\lambda)\, \tilde{F}_{2,h}(\tau'/\lambda - \tau/\lambda) + (c/\Lambda)\, \tilde{T}_{-1,h}(\lambda\tau, \tau_b)\, \tilde{T}_{-1,h}(\tau_a/\lambda, \tau')\, \tilde{F}_{2,h}(\lambda\tau - \lambda\tau')\} \qquad (79)$$

Finally, the expression for $\tilde{E}_h$ reads

$$\begin{aligned}\tilde{E}_h(\tau_a, \tau_b) = \int_0^{\infty} d\tau\, d\tau' \{ &(a\Lambda)[\tilde{F}_{2,h}(\lambda\tau)\, \tilde{T}_{1,h}(\tau_a, \tau'/\lambda)\, \tilde{T}_{-1,h}(\lambda\tau_b - \tau', \tau - \tau') \\ &+ \tilde{F}_{2,h}(\tau)\, \tilde{T}_{1,h}(\tau', \lambda\tau_b)\, \tilde{T}_{0,h}(\tau_a - \tau'/\lambda, \tau/\lambda - \tau'/\lambda)] \\ &+ (c/\Lambda)[\tilde{F}_{2,h}(\tau)\, \tilde{T}_{-1,h}(\tau_a/\lambda, \tau')\, \tilde{T}_{0,h}(\lambda\tau - \lambda\tau', \tau_b - \lambda\tau') \\ &+ \tilde{F}_{2,h}(\tau/\lambda)\, \tilde{T}_{-1,h}(\lambda\tau', \tau_b)\, \tilde{T}_{1,h}(\tau - \tau', \tau_a/\lambda - \tau')]\}\end{aligned} \qquad (80)$$

We note that Eq. (76) provides the equation of motion of $\tilde{T}_{0,h}$ only, whereas all the equations written above depend also on $\tilde{T}_{1,h}$ and $\tilde{T}_{-1,h}$. Equations of motion for these objects can be written similarly. We will however opt in the context of this first demonstration to disregard the differences between these functions; correspondingly we replace (without further ado) the last two functions with the first. Obviously this is an arbitrary step that is done for the sake of simplicity, and can (and will) be improved upon in later work. After this step we have a closed nonlinear set of integro-differential equations in terms of $\tilde{F}_{2,h}(\tau)$ and $\tilde{T}_{0,h}(\tau, \tau')$.

## 7. EXPLICIT DEMONSTRATION OF THE SOLVABILITY CONDITIONS

Given a set of integro-differential equations, it is customary to expand the unknown functions in an appropriate complete set of functions, and to project the resulting expanded form on each function in the set separately. In this way one generates a countable set of algebraic equations. The least automatic step in this procedure is the choice of the complete set of functions. By choosing the low order functions in the basis set to represent in some sense the properties of the expected solutions, one can hope that a truncated set may serve as well. It is of course also very helpful if the used set of functions is simple enough to allow as much analytic integration as possible. To find a good compromise between these requirements we analyzed the asymptotics (in time) of the equations and discovered that $\tilde{F}_{2,h}(\tau)$ decays like $e^{-|\tau|^x}$ with $x$ a function of $h$. For the sake of analytic integrations we decided to expand in linear combinations of generalized Laguerre functions

$$\mathscr{L}_n(\tau) \equiv L_n(\tau) \exp(-\tau^2) \tag{81}$$

with $L_n$ being generalized Laguerre polynomials, $L_0 = 1$, $L_1 = 1 - \sqrt{2\pi}\,\tau$, etc. These functions are orthogonal for integrations in the semi-infinite interval $[0, \infty]$, and our triplex is defined in this interval. The 2nd order correlation function is symmetric in $\tau$, and since we expect that our functions decay on an "$h$-slice" faster than any power of time with a characteristic time scale that has been already scaled out, the first contribution to the correlation function is

$$\begin{aligned} \tilde{F}_{2,h}(\tau) &= \mathscr{X} \mathscr{L}_0(|\tau|) + \text{h.o.t.} \\ &= \mathscr{X} \exp(-|\tau|^2) + \text{h.o.t} \end{aligned} \tag{82}$$

where "h.o.t" stand for higher order terms in this systematic expansion. The dimensionless coefficient $\mathscr{X}$ can be determined from the solution of the coupled equations. The triplex is zero for negative times, and an analysis of its equation reveals that it vanishes whenever at least one its arguments is zero. The first term in its expansion is thus proportional to

$$\tilde{\mathscr{L}}_1 \equiv \mathscr{L}_0 - \mathscr{L}_1 \propto \tau \exp(-\tau^2) \tag{83}$$

Accordingly the triplex is represented as

$$\begin{aligned} \tilde{T}_{0,h}(\tau, \tau') &= \mathscr{Y} \tilde{\mathscr{L}}_1(\tau)\, \tilde{\mathscr{L}}_1(\tau') + \text{h.o.t.} \\ &= \mathscr{Y} \tau\tau' \exp(-\tau^2 - \tau'^2) + \text{h.o.t} \end{aligned} \tag{84}$$

In principle we need now to introduce the expansion into our equations of motion, and then project each equation onto appropriate basis functions. For the sake of the present demonstration we will employ just the first term in the expansion as displayed in Eqs. (82) and (84). After this (significant) simplification (that can and will be improved in future work), we symmetrized the equation of $F_{2,h}$ with respect to $\tau \to -\tau$, and integrated the two final equations against $\mathscr{L}_0$ on the interval $[0, \infty]$.

The resulting equation which is inherited from Eq. (73) reads:

$$-\mathscr{X} + \mathscr{X}^2 \mathscr{Y} \left\{ a\Lambda V_a(\lambda) + bV_b(\lambda) + \frac{c}{\Lambda} V_c(\lambda) \right\} = 0 \tag{85}$$

where

$$\mathscr{X}^2\mathscr{Y}V_a(\lambda)=\int_{-\infty}^{\infty}d\tau\exp(-\tau^2)\,\tilde{F}_{3,h}(0,\lambda\tau,\lambda\tau) \tag{86}$$

$$\mathscr{X}^2\mathscr{Y}V_b(\lambda)=\int_{-\infty}^{\infty}d\tau\exp(-\tau^2)\,\tilde{F}_{3,h}(\tau,0,\tau) \tag{87}$$

$$\mathscr{X}^2\mathscr{Y}V_c(\lambda)=\int_{-\infty}^{\infty}d\tau\exp(-\tau^2)\,\tilde{F}_{3,h}\left(0,\frac{\tau}{\lambda},\frac{\tau}{\lambda}\right) \tag{88}$$

and the symmetrization results in the domain of integration $[-\infty,\infty]$. Note that all three $V(\lambda)$ functions depend on $\lambda$ through the dependence of $\tilde{F}_{3,h}$ on $\tilde{F}_{2,h}$, cf. Eq. (75). On the other hand they are independent of $\mathscr{X}$ and $\mathscr{Y}$. In a similar way, starting from Eq. (76) we derive the following equation:

$$\mathscr{Y}+\mathscr{X}\mathscr{Y}^2\left\{a\Lambda W_a(\lambda)+bW_b(\lambda)+\frac{c}{\Lambda}W_c(\lambda)\right\}=0 \tag{89}$$

where the functions $W_a(\lambda)$, $W_b(\lambda)$ and $W_c(\lambda)$ can be represented as integrals. They are too involved to warrant writing them explicitly, but we offer a graphical representation of the six functions $V(\lambda)$ and $W(\lambda)$ in Fig. 9.

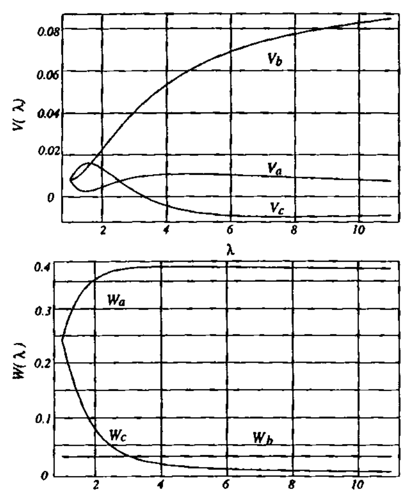


Fig. 9. The functions determining the solvability condition (89). Panel a: Graphic representation of the functions $V_a(\lambda)$, $V_b(\lambda)$ and $V_c(\lambda)$. Panel b: Graphic representation of the functions $W_a(\lambda)$, $W_b(\lambda)$ and $W_c(\lambda)$.

Equations (85) and (89) for the two unknowns $\mathscr{X}$ and $\mathscr{Y}$ are over-determined, and can actually fix only the product $\mathscr{X}\mathscr{Y}$. The solvability condition is now obvious: we need to require that $\mathscr{X}\mathscr{Y}$ computed from the first equation agrees with $\mathscr{X}\mathscr{Y}$ computed from the second. The condition is

$$a\Lambda U_a(\lambda) + bU_b(\lambda) + \frac{c}{\Lambda} U_c(\lambda) = 0 \tag{90}$$

where $U_a(\lambda) = V_a(\lambda) + W_a(\lambda)$, etc. This is a quadratic equation for $\Lambda$ whose solution is

$$\begin{aligned} \Lambda &\equiv q^{1-3h-\mathscr{Z}(h)} \\ &= [-bU_b(\lambda) \pm \sqrt{b^2U_b^2(\lambda) - 4acU_a(\lambda)\,U_c(\lambda)}]/2aU_a(\lambda) \end{aligned} \tag{91}$$

Given a value of $q$, the functions $U(\lambda)$ can be considered as functions of $h$ due to Eq. (74). The function $\mathscr{Z}(h)$ is now given by Eq. (91). We thus reached our goal of finding $\mathscr{Z}(h)$ for any given value of $q$, $a$, $b$ and $c$. We remember however that all these parameters are arbitrary, and we need to make an intelligent choice of their numerical values.

As we discussed before, the coefficients $a$, $b$ and $c$ are related, and only one of them is free, and we choose $a = 1$, $c = -a - b$ and $b$ remains as one of our freedoms. The second freedom is the value of $q$. We can remove these freedoms altogether by requiring that

$$\min_{h \geqslant 0} \mathscr{Z}(h) = 0, \qquad \text{implying} \quad \zeta_0 = 0 \tag{92}$$

$$\min_{h \geqslant 0} \{\mathscr{Z}(h) + 3h\} = 1, \qquad \text{implying} \quad \zeta_3 = 1 \tag{93}$$

With these two requirements the values of $q$ and $b$ are determined as $q = 10.23$ and $b = -3.20$. For these values of the parameters the function $\mathscr{Z}(h)$ exists in the interval $0 \leqslant h \leqslant h_{\max} = 0.462$, and is shown in Fig. 10. In the same figure we also display the functions $\mathscr{Z}(h) + h$ and $\mathscr{Z}(h) + 2h$, in full in the upper panel, and close to their minima in the lower panels. The vertical dashed line in the upper panel represents $h_{\max}$ where $\mathscr{Z}(h)$ has infinite derivative. From the values of the minima we can read an estimate for $\zeta_1 \approx 0.421$ and $\zeta_2 \approx 0.806$. Needless to say, we do not ascribe any profound meaning to these numbers, except for the two crucial facts: (i) *The exponents are not K41* and (ii) *The exponents are not linear function of n. In particular we note that* $2\zeta_1 \approx 0.842$ *which is larger than* $\zeta_2$. This is the only reason to report the values of the exponents to three digits.

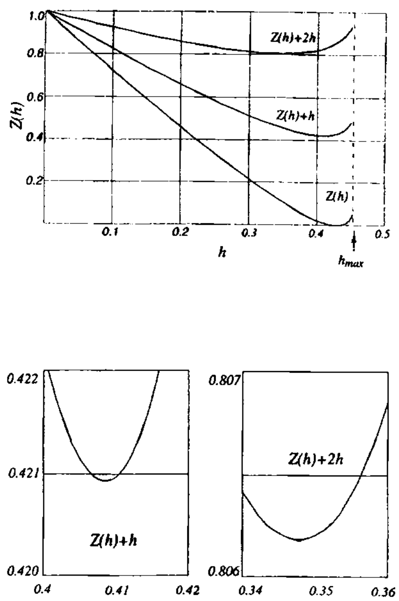


Fig. 10. Upper panel: The functions $\mathscr{Z}(h)+nh$ with $n=0$, 1 and 2, for $q=10.23$ and $b=-3.20$. Lower panels: Blow ups of the region near the minimum of $\mathscr{Z}(h)+h$ and $\mathscr{Z}(h)+2h$. The position of the minima determines the scaling exponents $\zeta_1 \approx 0.421$ and $\zeta_2 \approx 0.806$. Note that this demonstrates multiscaling.

In addition, we observe that $\mathscr{Z}(h)$ is positive throughout its range. This means that the saddle point integral (14) is computed with $L$ as the largest scale in the problem; this identifies the outer scale of turbulence as the renormalization scale. This is an important result of the analysis which is independent of the numerical values of the exponents.

## 8. SUMMARY AND DISCUSSION

### A. Summary of the Approximations

In our procedure we committed six crimes against precision. The first was closing the hierarchy at the level of the fourth-order objects. Consequently we can hope to get nonlinear dependence of $\zeta_n$ on $n$ only up to $n=3$. Beyond this we must construct higher order correlation functions in terms of the objects at hand, and as a result the scaling exponents are estimated as linear extrapolations:

$$\zeta_n = \zeta_3 + (n-3)(\zeta_3 - \zeta_2), \qquad n > 3 \tag{94}$$

This approximation can be improved by closing the hierarchy at a higher $n = m$. In that case one would find nonlinear $\zeta_n$ up to that value of $n = m$, and above it a linear extrapolation

$$\zeta_n = \zeta_m + (n-m)(\zeta_n - \zeta_{n-1}), \qquad n > m \tag{95}$$

If, as is believed phenomenologically, the exponents become asymptotically a linear function of $n$, then our procedure will converge with increasing $m$. This means in fact that there exists a numerical small parameter related to $1/m$.

Within the closure discussed in this paper ($m = 3$) we made the additional approximation of neglecting the two vertices $B$ and $C$. This step can be avoided at the price of increasing the complexity three-fold.

At this point we disregarded (for the sake of simplicity) the tensor indices of all the objects of the theory. This "scalar" approximation reduces the complexity significantly.

The next approximation involved treating the space integrals as a 3-scale estimate. To improve upon this approximation one needs to expand the spatial dependence of the statistical objects in a complete set of functions in space *and* time. This sounds frightening, but in fact one can again use a truncated expansion using clever asymptotic.

Within the 3-scale approximation we simplified further neglecting the difference between the three different triplexes. It is relatively easy to relax this step of approximation.

Finally, we truncated severely these set of functions representing the time dependence of our objects. Any addition of basis functions increases the complexity of the equations significantly.

All these steps are uncontrolled, but we know in principle how to improve them. It will take some effort to examine which of the approximations is most crucial, and how to achieve more tight calculations. Nevertheless we believe that we have demonstrated convincingly that our procedure produces in principle anomalous exponents that are nonlinear function of $n$. In addition we showed how the outer scale of turbulence is chosen as the renormalization scale. These important results encourage us to improve upon the procedure described above to obtain more accurate numerics.

## B. Comparison to Kraichnan's Model of Turbulent Advection

A model that has drawn considerable attention to the issue of anomalous scaling in turbulent fields is Kraichnan's model of turbulent advection, in which a passive scalar is adverted by a velocity field which is scaling in space but delta-correlated in time.[7] We refer the reader to the literature on this problem[12–16] to get the full background for the following comments. We think that it may be instructive to compare the way that anomalous scaling appears in that and the present problem to gain more insight about the fundamental differences between the two.

The problem of passive scalar advection is *linear* in the advected field. In addition, the choice of a rapidly varying velocity field with delta-correlation in time allows the development of a theory for the same-time fully-unfused correlations functions $\mathscr{T}_n$ (in which the velocity field is replaced by the scalar field). Kraichnan showed that such functions satisfy exact differential equations of the form

$$\mathscr{B}_{2n}\mathscr{T}_{2n} = \mathrm{RHS}(\mathscr{T}_{2n-2}) \tag{96}$$

where $\mathscr{B}_n$ is a differential operator. Attempting to find the scaling exponents from this equation by power counting results in linear scaling in which $\zeta_{2n} = n\zeta_2$. It was pointed out however[13] that the solutions of the associated *homogeneous* equation $\mathscr{B}_{2n}\mathscr{T}_{2n} = 0$ can be much larger than the inhomogeneous solutions of (96), and these are free of power counting. Indeed, these "zero modes" revealed anomalous exponents, and the renormalization scale appeared through satisfying the boundary conditions at large scales. Since the equation for every order $n$ of $\mathscr{T}_{2n}$ is decoupled from the rest, it is not awfully surprising that the scaling exponents for different $n$ are not simply related.

In contradistinction, the problem of hydrodynamic turbulence is *nonlinear* and there is no way to derive a theory in terms of simultaneous correlation functions $\mathscr{T}_n$. We are bound to develop a theory in terms of the space-time correlation functions $\mathscr{F}_n$. These objects were shown explicitly to break dynamical scaling, and as their equations are coupled, it was much less obvious how to achieve a theory that is free of power-counting. We discovered however that in the limit of high Reynolds numbers the exact equations for the fully-unfused correlation functions exhibit rescaling symmetry that can be used to advantage. By focusing on the equation of motion in an "*h*-slice" we could develop a theory in which the all-important function $\mathscr{Z}(h)$ becomes a generalized eigenvalue. It then became crucial, in the course of closure and approximations, not to destroy the fundamental rescaling symmetry, to allow for the presence of new scaling solutions with anomalous scaling exponents. The renormalization scale appears as a result of computing saddle point integrals.

In spite of the great differences between the two problems, the common point is that in order to find anomalous scaling solutions we must reach equations that are neutral to rescaling, in which power counting plays no role. If we obtain such equations, the rest is technicalities, which may be more or less formidable, but they are just technicalities.

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It is a pleasure to dedicate this paper to Leo Kadanoff on the occasion of his 60th birthday. We hope that the reappearance of the old $f(\alpha)$ multifractal formalism in a new guise in the problem of turbulence will be an appropriate tribute to Leo's far-sightedness regarding the importance of multifractals and their relevance to turbulence. Leo, we wish you many years of exciting work.

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